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EXTREME POINTS AND RETRACTIONS IN BANACH SPACES

BY

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ABSTRACT

For T a completely regular topological space and X a strictly convex Banach space, we study the extremal structure of the unit ball of the space $C(T,X)$ of continuous and bounded functions from T into X. We show that when $\dim X$ is an even integer then every point in the unit ball of $C(T,X)$ can be expressed as the average of three extreme points if, and only if, $\dim T < \dim X$, where $\dim T$ is the covering dimension of T. We also prove that, if X is infinite-dimensional, the aforementioned representation of the points in the unit ball of $C(T,X)$ is always possible without restrictions on the topological space T. Finally, we deduce from the above result that the identity mapping on the unit ball of an infinite-dimensional strictly convex Banach space admits a representation as the mean of three retractions of the unit ball onto the unit sphere.

Let Y be a normed space; we will denote by $B(Y)$, $S(Y)$ and $E(Y)$ its closed unit ball, its unit sphere and the set of extreme points of $B(Y)$, respectively.

The aim of this paper is to obtain new information about a conjecture posed by Cantwell (see [3]) about the extremal structure of the unit ball in spaces of continuous functions. Namely, results of [1], [3], [5], [6] and [7] are extended.

From now on, T denotes a topological space, X a real normed space and $C(T,X)$ the space of continuous and bounded functions from T into X with its

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usual uniform norm. Frequently we will write Y instead of $C(T,X)$. Topological spaces in this note are assumed to be completely regular*. In view of [10, Corollary 1.8] this supposition is scarcely restrictive.

As we have already said, we prove in this note that if X is a strictly convex normed space with even dimension, every element of the unit ball of $C(T,X)$ can be expressed as the average of three extreme points if, and only if, $\dim T < \dim X$, where dim T denotes the covering dimension of T (see [4] for definitions). This number 3 cannot be improved in general [8]. We also show that, when X is an infinite-dimensional strictly convex Banach space and T is any topological space, the aforementioned representation of the points in the unit ball of $C(T,X)$ is always possible. From this result we deduce that the identity mapping on the unit ball of an infinite-dimensional strictly convex Banach space is the average of three retractions of the unit ball onto the unit sphere.

It is worth mentioning that our results depend on the existence of continuous mappings from the unit sphere of a normed space into itself without fixed or antipodal points. This excludes the spaces of odd dimension. The existence of such mappings for infinite-dimensional Banach spaces was proved in [1, Proposition 12].

To get our objectives we will need the following lemma which can be proved similarly to [7, Lemma 3] (which is a particular case of it).

LEMMA 1: *Let M be a two-dimensional strictly convex normed space. Consider a in $M \setminus \{0\}$ and $f \in M^*$ such that $\ker f = \mathrm{Lin}\,\{a\}$ (the linear expansion of $\{a\}$). Suppose there exist $x, y \in S(M)$ satisfying $f(x) \geq 0$, $f(y) \geq 0$ and $\|x - a\| = \|y - a\|$. Then $x = y$.*

Let M be a normed space; given $b \in M$ and $\delta > 0$ we denote

$$S(b,\delta) = \{x \in M \colon \|x - b\| = \delta\}.$$

The geometric meaning of Lemma 1 is clarified by the following Proposition (which is an immediate consequence of it).

PROPOSITION 2: *Let M satisfy the hypotheses of Lemma 1. Then, given $b_1, b_2 \in M$ with $b_1 \neq b_2$ and $\delta_1, \delta_2 \in \mathbb{R}^+$, the set*

$$S(b_1,\delta_1) \cap S(b_2,\delta_2)$$

* The results extend to all topological spaces if $\dim T < \dim X$ is replaced by some technical condition (see in [1] the extension property).

contains at most two points.

The next result is a generalization of [1, Proposition 9] which can be obtained by taking $\lambda = \frac{1}{2}$.

PROPOSITION 3: *Let X be a strictly convex normed space. Assume there exists a continuous mapping $v\colon S(X) \to S(X)$ with $v(x) \neq x$, $v(x) \neq -x$, for every x in $S(X)$. Given any $\lambda \in]0,1[$ consider the set*

$$B = \{x \in X \setminus \{0\} \colon |2\lambda - 1| \leq \|x\| \leq 1\}.$$

Then there are continuous mappings $\Phi_1, \Phi_2\colon B \to S(X)$ such that

$$x = \lambda\Phi_1(x) + (1-\lambda)\Phi_2(x), \quad \forall x \in B.$$

Proof: Define $g\colon [0,2] \times B \to X$ by

$$g(t,x) = \begin{cases} (1-t)\frac{x}{\|x\|} + tv\left(\frac{x}{\|x\|}\right) & \text{if} \quad 0 \leq t \leq 1, \\ \\ (2-t)v(\frac{x}{\|x\|}) - (t-1)\frac{x}{\|x\|} & \text{if} \quad 1 \leq t \leq 2. \end{cases}$$

Evidently g is continuous and $g(t,x) \neq 0, \forall (t,x) \in [0,2] \times B$. So we can consider the mapping $\Gamma\colon [0,2] \times B \to S(X)$ defined by $\Gamma(t,x) = \frac{g(t,x)}{\|g(t,x)\|}$, $\forall (t,x) \in [0,2] \times B$.

Let us fix $x \in B$. It is clear that $2\lambda - 1 \leq \|x\| \leq 1$, hence $|\|x\| - \lambda| \leq 1 - \lambda$ and so

$$\left\| \frac{x}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(0,x) \right\| = \frac{1}{1-\lambda}\left\| x - \lambda\frac{x}{\|x\|} \right\| = \frac{|\|x\| - \lambda|}{1-\lambda} \leq 1.$$

On the other hand (taking into account that $\|x\| \geq 1 - 2\lambda$)

$$\left\| \frac{x}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(2,x) \right\| = \frac{1}{1-\lambda}\left\| x + \lambda\frac{x}{\|x\|} \right\| = \frac{\|x\| + \lambda}{1-\lambda} \geq 1.$$

Consequently, there is $t \in [0,2]$ such that

$$(*) \qquad \left\| \frac{x}{1-\lambda} - \frac{\lambda}{1-\lambda}\Gamma(t,x) \right\| = 1,$$

that is, $\left\| \frac{x}{\lambda} - \Gamma(t,x) \right\| = \frac{1-\lambda}{\lambda}$. Let $a = \frac{x}{\lambda}$, $M = \operatorname{Lin}\left\{a, v\left(\frac{x}{\|x\|}\right)\right\}$ and consider $f \in M^*$ such that $\ker f = \operatorname{Lin}\{a\}$ and $f\left(v\left(\frac{x}{\|x\|}\right)\right) > 0$. Then it is clear that $f(\Gamma(t,x)) \geq 0, \forall t \in [0,2]$. Moreover it is easy to check that if $t_1, t_2 \in [0,2]$ and $t_1 \neq t_2$ then $\Gamma(t_1,x) \neq \Gamma(t_2,x)$. Thus by Lemma 1 there is only one t for which $(*)$ holds. We denote it by $t(x)$.

The mapping $x \longmapsto t(x)$ from B into $[0,2]$ is continuous. Indeed, if $x \in B$ is a point of discontinuity of the above mapping, then we can find a sequence $\{x_n\}$ of elements in B and t in $[0,2]$ such that $\{x_n\} \to x$ and $\{t(x_n)\} \to t \neq t(x)$. By the continuity of Γ we can assert that $\left\|\frac{x-\lambda\Gamma(t,x)}{1-\lambda}\right\| = 1$ and this contradicts the unicity of $t(x)$.

Finally, define $\Phi_1, \Phi_2\colon B \to S(X)$ by

$$\Phi_1(x) = \Gamma(t(x),x), \quad \Phi_2(x) = \frac{x - \lambda\Gamma(t(x),x)}{1-\lambda}, \quad \forall x \in B.$$

It is obvious that Φ_1 and Φ_2 satisfy the required conditions. ∎

It is well known that if X is a Banach space with even dimension there is a continuous mapping $v\colon S(X) \to S(X)$ such that $v(x) \neq x$ and $v(x) \neq -x$, for every x in $S(X)$. The existence of such mappings for infinite-dimensional Banach spaces was proved in [1, Proposition 12].

We can now prove the following result.

THEOREM 4: *Let T be a topological space and X a strictly convex Banach space with even or infinite dimension. If $Y = C(T,X)$, given $f \in B(Y)$ with $f(t) \neq 0, \forall t \in T$ and $\lambda \in [\frac{1}{2}, \frac{1}{2}(1+m(f))]$ (where $m(f) = \operatorname{Inf}\{\|f(t)\|: t \in T\}$) there exist $e_1, e_2 \in E(Y)$ such that*

$$f = \lambda e_1 + (1-\lambda)e_2.$$

Proof: Let f and λ satisfy the above assumptions. Then it is clear that $m(f) \geq 2\lambda - 1$, hence $\|f(t)\| \geq 2\lambda - 1 = |2\lambda - 1|, \forall t \in T$. If $\lambda = 1$ it is obvious that $f \in E(Y)$ and we can consider $e_1 = e_2 = f$. If $\lambda < 1$ let's choose Φ_1 and Φ_2 verifying the conditions of the above proposition. Then

$$f(t) = \lambda\Phi_1(f(t)) + (1-\lambda)\Phi_2(f(t)), \quad \forall t \in T.$$

So the proof is completed by defining $e_1 = \Phi_1 \circ f$ and $e_2 = \Phi_2 \circ f$. ∎

In order to obtain our main result we need the following proposition which was proved in [2, Proposition 5.2] in case Y is a C^*-algebra.

PROPOSITION 5: *Under the hypotheses of Theorem 4 let's fix $e \in E(Y)$, $g \in B(Y)$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ with $\alpha > \beta$, $\alpha + \beta = \gamma + \delta$ and γ, δ in $[\beta, \alpha]$. Then there exist $e_1, e_2 \in E(Y)$ such that $\alpha e + \beta g = \gamma e_1 + \delta e_2$.*

Proof: Clearly the mapping $f = \frac{\alpha e+\beta g}{\alpha+\beta}$ omits the origin. In fact,

$$m(f) \geq \frac{\alpha-\beta}{\alpha+\beta} > 0.$$

Let's suppose without loss of generality that $\delta \leq \gamma$. If we define $\lambda = \frac{\gamma}{\alpha+\beta}$ it is easily seen that $\frac{1}{2} \leq \lambda \leq \frac{1}{2}(1+m(f))$. So, by the above theorem, there exist $e_1, e_2 \in E(Y)$ such that $f = \lambda e_1 + (1-\lambda)e_2$, that is $\frac{\alpha e+\beta g}{\alpha+\beta} = \frac{\gamma}{\alpha+\beta}e_1 + \frac{\delta}{\alpha+\beta}e_2$. ∎

THEOREM 6: *Let T be a topological space, X a strictly convex Banach space and $Y = C(T, X)$. Suppose that either of the following conditions hold:*

1. *X has even dimension and* $\dim T < \dim X$ *or*
2. *X is infinite-dimensional.*

Then, given $f \in B(Y)$ and $\lambda_1, \lambda_2, \lambda_3 \in]0, \frac{1}{2}[$ with $\lambda_1+\lambda_2+\lambda_3 = 1$, there exist $e_1, e_2, e_3 \in E(Y)$ such that

$$f = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3.$$

Proof: We can naturally suppose that $\lambda_1 \geq \lambda_2$ and $\lambda_1 \geq \lambda_3$. For ϵ sufficiently small we have $\alpha := \lambda_1 + \epsilon < \frac{1}{2}$, $\beta := \lambda_2 - \epsilon > 0$ and, of course, $\alpha > \beta$ and $\alpha > \lambda_3$. By [1, Theorem 2 and Corollaries 4 and 5] there are $e \in E(Y)$ and $g \in B(Y)$ such that

$$f = \alpha e + (1-\alpha)g = \alpha e + \beta g + \lambda_3 g.$$

Now, by Proposition 5 there exist $e_1', e_2' \in E(Y)$ such that $\alpha e + \beta g = \alpha e_1' + \beta e_2'$. Hence $f = \alpha e_1' + \beta e_2' + \lambda_3 g$. For the same reason there are $e_1'', e_3 \in E(Y)$ for which $\alpha e_1' + \lambda_3 g = \alpha e_1'' + \lambda_3 e_3$ and so $f = \alpha e_1'' + \beta e_2' + \lambda_3 e_3$. Finally, since $\alpha + \beta = \lambda_1 + \lambda_2$ and $\lambda_1, \lambda_2 \in [\beta, \alpha]$ we can again apply the above proposition to obtain $e_1, e_2 \in E(Y)$ such that $\alpha e_1'' + \beta e_2' = \lambda_1 e_1 + \lambda_2 e_2$. Therefore $f = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ and the proof is complete. ∎

From [2, Theorem 5.7] and [9, Corollaries 3.6 and 3.7] a similar result can be derived for $Y = C(T, \mathbb{C})$.

Remark 7: It is easy to check that if Y is any normed space satisfying the conclusion of the above Theorem, then Y satisfies, in fact, the following property:

$$B(Y) = \lambda_1 E(Y) + \lambda_2 E(Y) + \cdots + \lambda_k E(Y)$$

for every $k \geq 3$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in]0, \frac{1}{2}[$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. ∎

The infinite-dimensional case of Theorem 6 is trivially equivalent to the following result which is obtained by taking $T = B(X)$.

COROLLARY 8: *Let X be an infinite-dimensional strictly convex Banach space. Then, for each natural $k \geq 3$ and $\lambda_1, \lambda_2, \dots, \lambda_k \in]0, \frac{1}{2}[$ with $\lambda_1 + \lambda_2 + \dots + \lambda_k = 1$, there exist retractions $e_1, e_2, \dots, e_k$ of the unit ball of X onto the unit sphere of X such that*

$$x = \lambda_1 e_1(x) + \lambda_2 e_2(x) + \cdots + \lambda_k e_k(x), \qquad \text{for every } x \text{ in } B(X).$$

In particular, there exist three retractions e_1, e_2, e_3 from $B(X)$ onto $S(X)$ such that

$$x = \frac{e_1(x) + e_2(x) + e_3(x)}{3}, \quad \forall x \in B(X).$$

The next result shows that the number 3 in the previous Corollary is the best possible. First let us observe that if $\lambda \in]0, 1[$ and e_1, e_2 are continuous mappings from $B(X)$ into $S(X)$ with

$$x = \lambda e_1(x) + (1 - \lambda) e_2(x), \quad \forall x \in B(X),$$

then $\lambda = 1 - \lambda = \frac{1}{2}$.

PROPOSITION 9: *Let X be an arbitrary normed space. There are no two continuous mappings e_1, e_2 from $B(X)$ into $S(X)$ such that*

$$x = \frac{e_1(x) + e_2(x)}{2} \qquad \text{for every } x \text{ in } B(X).$$

Proof: Suppose there exist e_1 and e_2 satisfying the above conditions and let $u = e_1(0)$. Given $t \in]0, \frac{1}{2}[$, we have

$$\begin{aligned} \|u - e_1(tu)\| &= \left\| \frac{1}{t} tu - e_1(tu) \right\| = \left\| \frac{1}{t} \frac{e_1(tu) + e_2(tu)}{2} - e_1(tu) \right\| \\ &= \left\| \frac{(1 - 2t) e_1(tu) + e_2(tu)}{2t} \right\| \geq \frac{1 - (1 - 2t)}{2t} = 1. \end{aligned}$$

By continuity of e_1, letting t tend to 0, we get $\|u - e_1(0)\| \geq 1$, a contradiction. ∎

We now suppose that X is finite-dimensional. It was proved in [5, Corollary 9] that if $\dim X \geq 2$ and T is a completely regular space, then $B(Y) = \mathrm{co}(E(Y))$ if, and only if, $\dim T < \dim X$, where $\mathrm{co}(E(Y))$ denotes the convex hull of $E(Y)$. Accordingly, the following result is clear in view of Theorem 6 and Remark 7.

COROLLARY 10: *Let T be a topological space and X a strictly convex space with even dimension. For $Y = C(T, X)$ the following conditions are equivalent:*

1. *For every $\lambda_1, \lambda_2, \lambda_3 \in]0, \frac{1}{2}[$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$,*

$$B(Y) = \lambda_1 E(Y) + \lambda_2 E(Y) + \lambda_3 E(Y).$$

2. $B(Y) = \frac{E(Y)+E(Y)+\overset{(k)}{\cdots}+E(Y)}{k}$ *for every $k \geq 3$.*
3. $B(Y) = \mathrm{co}(E(Y))$ *(the convex hull of $E(Y)$).*
4. $\dim T < \dim X$.

The equivalence between conditions 3 and 4 in the above result was proved in [1, Corollary 11] but condition 2 was only obtained for $k = 4$. Moreover, as we have already said, in [5, Corollary 9] were also proved the aforementioned equivalence for $\dim X \geq 2$ (even or odd) but in this (more general) case condition 2 was obtained for $k = 8$. Other particular cases can be found in [3] and [7].

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